HILBERT-KUNZ MULTIPLICITY OF THREE-DIMENSIONAL LOCAL RINGS

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Abstract. In this paper, we investigate a lower bound (say $s_{\mathrm{HK}}(p,d)$) on Hilbert-Kunz multiplicities for non-regular unmixed local rings of Krull dimension d with characteristic $p>0$. Especially, we focus three-dimensional local rings. In fact, as a main result, we will prove that $s_{\mathrm{HK}}(p,3)=4/3$ and that a three-dimensional complete local ring of Hilbert-Kunz multiplicity $4/3$ is isomorphic to the non-degenerate quadric hyperplanes $k[[X,Y,Z,W]]/(X^2+Y^2+Z^2+W^2)$ under mild conditions.

Furthermore, we pose a generalization of the main theorem to the case of $\dim A \geq 4$ as a conjecture, and show that it is also true in case of $\dim A = 4$ using the similar method as in the proof of the main theorem.

Introduction

Let A be a commutative Noetherian ring of characteristic $p>0$ with unity. In [15], Kunz proved the following theorem, which gives a characterization of regular local rings of positive characteristic.

Kunz' Theorem ([15]). Let $(A,\mathfrak{m},k)$ be a local ring of characteristic $p>0$. Then the following conditions are equivalent:

1. A is a regular local ring.
2. A is reduced and is flat over a subring $A^p=\{a^p \mid a\in A\}$. In other words, the Frobenius map $F: A\to A$ $(a\mapsto a^p)$ is flat.
3. $l_A(A/\mathfrak{m}^{[q]})=q^d$ for any $q=p^e, e\geq 1$, where $\mathfrak{m}^{[q]}=(a^q \mid a\in\mathfrak{m})$ and $l_A(M)$ denotes the length of an A-module M.

Furthermore, in [16], Kunz observed that $l_A(A/\mathfrak{m}^{[q]})/q^d$ $(q=p^e)$ is a reasonable measure for singularity of a local ring. Based on the idea of Kunz, Monsky [18] proved that there exists a constant $c=c(A)$ such that

$$l_A(A/\mathfrak{m}^{[q]})=cq^d+O(q^{d-1})$$

and defined the notion of *Hilbert-Kunz multiplicity* by $e_{\mathrm{HK}}(A)=c$. In 1990's, Han and Monsky [10] have given an algorism to compute the Hilbert-Kunz multiplicity for any hypersurface of Briskorn–Fermat type

$$A=k[X_0,\dots,X_n]/(X_0^{d_0}+\cdots+X_n^{d_n}).$$

See e.g. [1, 2, 4, 24] about the other examples. Hochster and Huneke [11] have given "Length Criterion for Tight Closure" in terms of Hilbert-Kunz multiplcity

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(see Theorem 1.8) and indicated the close relation between the tight closure and the Hilbert-Kunz multiplicity. In [22], the authors have proved the theorem which gives a characterization of regular local rings in terms of Hilbert-Kunz multiplicity as follows:

Theorem A ([22, Theorem (1.5)])**.** Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of positive characteristic. Then A is regular if and only if $e_{\mathrm{HK}}(A) = 1$.

Many researchers have tried to improve this theorem. For example, Blickle and Enescu [3] recently proved the following theorem:

Theorem B ([3])**.** Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of characteristic $p > 0$. Then the following statements hold:

(1) If $e_{\mathrm{HK}}(A) < 1 + \frac{1}{d!}$, then A is Cohen–Macaulay and F-rational.
(2) If $e_{\mathrm{HK}}(A) < 1 + \frac{1}{p^d d!}$, then A is regular.

So it is natural to consider the following problem.

Problem C. Let $d \geq 2$ be any integer. Determine the value

$$s_{\mathrm{HK}}(p, d) := \inf\{e_{\mathrm{HK}}(A) \mid A \text{ is a non-regular unmixed local ring of } \operatorname{char} A = p\}.$$

Also, characterize the local rings A for which $e_{\mathrm{HK}}(A) = s_{\mathrm{HK}}(p, d)$ hold.

In case of one-dimensional local rings, it is easy to answer to this problem. In fact, $s_{\mathrm{HK}}(p, 1) = 2$; $e_{\mathrm{HK}}(A) = 2$ if and only if $e(A) = 2$. In case of two-dimensional Cohen–Macaulay local rings, the authors [23] have given a complete answer to this problem. Namely, we have $s_{\mathrm{HK}}(p, 2) = \frac{3}{2}$ by the theorem below.

Theorem D. (See also Corollary 2.5.) Let $(A, \mathfrak{m}, k)$ be a two-dimensional Cohen–Macaulay local ring of positive characteristic. Put $e = e(A)$, the multiplicity of A. Then the following statements hold:

(1) $e_{\mathrm{HK}}(A) \geq \frac{e+1}{2}$.
(2) Suppose that $k = \overline{k}$. Then $e_{\mathrm{HK}}(A) = \frac{e+1}{2}$ holds if and only if the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(A)$ is isomorphic to the Veronese subring $k[X, Y]^{(e)}$.

In the following, let us explain the organization of this paper. In Section 1, we recall the notions of Hilbert-Kunz multiplicity and tight closure etc. and gather several fundamental properties of them. In particular, Goto–Nakamura's theorem (Theorem 1.9) is important because it plays a central role in the proof of main result (Theorem 3.1) and the others.

In Section 2, we give a key result to estimate Hilbert-Kunz multiplicities for local rings in lower dimension. Indeed, Theorem 2.2 is a refinement of the argument in [23, Section 2]. Also, the point of our proof is to estimate $l_A(\mathfrak{m}^{[q]}/J^{[q]})$ (where J is a minimal reduction of $\mathfrak{m}$) using volumes in $\mathbb{R}^d$.

In Section 3, we prove the following theorem as the main result in this paper.

Theorem 3.1. Let $(A, \mathfrak{m}, k)$ be a three-dimensional unmixed local ring of characteristic $p > 0$. Then the following statements hold.

(1) If A is not regular, then $e_{\mathrm{HK}}(A) \geq \frac{4}{3}$.
(2) Suppose that $k = \overline{k}$ and $\operatorname{char} k \neq 2$. Then the following conditions are equivalent:
   (a) $e_{\mathrm{HK}}(A) = \frac{4}{3}$.

(b) $\widehat{A} \cong k[[X,Y,Z,W]]/(X^2+Y^2+Z^2+W^2)$.

Also, we study lower bounds on $e_{\mathrm{HK}}(A)$ for local rings A having a given (small) multiplicity e. In particular, we will prove that any three-dimensional unmixed local ring A with $e_{\mathrm{HK}}(A) < 2$ is F-rational.

In Section 4, we consider a generalization of Theorem 3.1 and pose the following conjecture.

Conjecure 4.2. Let $d \geq 1$ be an integer and $p > 2$ a prime number. Put

$$A_{p,d} := \overline{\mathbb{F}_p}[[x_0, x_1, \ldots, x_d]]/(x_0^2 + \cdots + x_d^2).$$

Let $(A, \mathfrak{m}, k)$ be a d-dimensional unmixed local ring with $k = \overline{\mathbb{F}_p}$. Then the following statements hold.

1. (1) If A is not regular, then $e_{\mathrm{HK}}(A) \geq e_{\mathrm{HK}}(A_{p,d}) \geq 1 + \frac{c_d}{d!}$ (See 4.2 for the definition of c_d). In particular, $s_{\mathrm{HK}}(p,d) = e_{\mathrm{HK}}(A_{p,d})$.
2. (2) If $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(A_{p,d})$, then the $\mathfrak{m}$-adic completion $\widehat{A}$ of A is isomorphic to $A_{p,d}$ as local rings.

Also, we prove that this is true in case of $\dim A = 4$. Namely we will prove the following theorem.

Theorem 4.3. Let $(A, \mathfrak{m}, k)$ be a four-dimensional unmixed local ring of characteristic $p > 0$. Also, suppose that $k = \overline{k}$ and $\operatorname{char} k \neq 2$. Then $e_{\mathrm{HK}}(A) \geq \frac{5}{4}$ if $e(A) \geq 3$, where $e(A)$ denotes the multiplicity of A. Also, the following statements hold.

(1) If A is not regular, then $e_{\mathrm{HK}}(A) \geq e_{\mathrm{HK}}(A_{p,4}) = \frac{29p^2+15}{24p^2+12}$.

(2) The following conditions are equivalent:
   - (a) Equality holds in (1).
   - (b) $e_{\mathrm{HK}}(A) < \frac{5}{4}$.
   - (c) $\widehat{A}$ is isomorphic to $A_{p,4}$.

1. Preliminaries

Throughout this paper, let A be a commutative Noetherian ring with unity. Furthermore, we assume that A has a positive characteristic p, that is, it contains a prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, unless specified. For every positive integer e, let $q = p^e$. If I is an ideal of A, then $I^{[q]} = (a^q \mid a \in I)A$. Also, we fix the following notation: $l_A(M)$ (resp. $\mu_A(M)$) denotes the length (resp. the minimal number of generators) of M for any finitely generated A-module M.

First, we recall the notion of Hilbert-Kunz multiplicity (see [15, 16, 18]), which plays a central role in this paper. Also, see [17] or [20] for usual multiplicity.

Definition 1.1 (Multiplicity, Hilbert-Kunz Multiplicity)**.** Let $(A, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ with $\dim A = d$. Let I be an $\mathfrak{m}$-primary ideal of A, and let M be a finitely generated A-module. The (*Hilbert-Samuel*) *multiplicity* $e(I, M)$ of I with respect to M is defined by

$$e(I, M) = \lim_{n \to \infty} \frac{d!}{n^d}\, l_A(M/I^n M)$$

The *Hilbert-Kunz multiplicity* $e_{\mathrm{HK}}(I, M)$ of I with respect to M is defined by

$$e_{\mathrm{HK}}(I, M) = \lim_{q \to \infty} \frac{l_A(M/I^{[q]}M)}{q^d}.$$

By definition, we put $e(I) = e(I, A)$ (resp. $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(I, A)$) and $e(A) = e(\mathfrak{m})$ (resp. $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(\mathfrak{m})$).

We recall several basic results on Hilbert-Kunz multiplicity.

Proposition 1.2 (Fundamental Properties (cf. [13, 15, 16, 18, 22]))**.** *Let $(A, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$. Let I, I' be $\mathfrak{m}$-primary ideals of A, and let M be a finitely generated A-module. Then the following statements hold.*

(1) *If $I \subseteq I'$, then $e_{\mathrm{HK}}(I) \geq e_{\mathrm{HK}}(I')$.*
(2) $e_{\mathrm{HK}}(A) \geq 1$.
(3) $\dim M < d$ *if and only if* $e_{\mathrm{HK}}(I, M) = 0$.
(4) *If $0 \to L \to M \to N \to 0$ is a short exact sequence of finitely generated A-modules, then*

$$e_{\mathrm{HK}}(I, M) = e_{\mathrm{HK}}(I, L) + e_{\mathrm{HK}}(I, N).$$

(5) (Associative Formula)

$$e_{\mathrm{HK}}(I, M) = \sum_{\mathfrak{p} \in \mathrm{Assh}\,(A)} e_{\mathrm{HK}}(I, A/\mathfrak{p}) \cdot l_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}),$$

where $\mathrm{Assh}\,(A)$ *denotes the set of minimal prime ideals* $\mathfrak{p}$ *of* A *with* $\dim A/\mathfrak{p} = \dim A$.
(6) *If J is a parameter ideal of A, then $e_{\mathrm{HK}}(J) = e(J)$. In particular, if J is a minimal reduction of I (i.e., J is a paramater ideal which is contained in I and $I^{r+1} = JI^r$ for some integer $r \geq 0$), then $e_{\mathrm{HK}}(J) = e(I)$.*
(7) *If A is regular, then $e_{\mathrm{HK}}(I) = l_A(A/I)$.*
(8) (Localization) $e_{\mathrm{HK}}(A_{\mathfrak{p}}) \leq e_{\mathrm{HK}}(A)$ *holds for any prime ideal* $\mathfrak{p}$ *such that* $\dim A/\mathfrak{p} + \mathrm{height}\,\mathfrak{p} = \dim A$.
(9) *If $x \in I$ is A-regular, then $e_{\mathrm{HK}}(I) \leq e_{\mathrm{HK}}(I/xA)$.*
(10) *If $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a flat local ring homomorphism such that $B/\mathfrak{m}B$ is a field, then $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(IB)$.*

Remark 1. Also, the similar result as above (except (6),(7)) holds for usual multiplicities.

Let $(A, \mathfrak{m}, k)$ be any local ring of positive dimension. *The associated graded ring* $\mathrm{gr}_{\mathfrak{m}}(A)$ of A with respect to $\mathfrak{m}$ is defined as follows:

$$\mathrm{gr}_{\mathfrak{m}}(A) := \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

Then $G = \mathrm{gr}_{\mathfrak{m}}(A)$ is a homogeneous k-algebra such that $\mathfrak{M} := G_+$ is the unique homogeneous maximal ideal of G. If $\mathrm{char}\, A = p > 0$ and $\dim A = d$, then $G_{\mathfrak{M}}$ is also a local ring of characteristic p with $\dim G_{\mathfrak{M}} = d$.

Proposition 1.3. ([22, Theorem (2.15)], [14]) *Let $(A, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$. Let $G = \mathrm{gr}_{\mathfrak{m}}(A)$ the associated graded ring of A with respect $\mathfrak{m}$ as above. Then $e_{\mathrm{HK}}(A) \leq e_{\mathrm{HK}}(G_{\mathfrak{M}}) \leq e(A)$.*

Remark 2. We use the same notation as in the above proposition. Although $e(A) = e(G_{\mathfrak{M}})$ always holds, $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(G_{\mathfrak{M}})$ seldom holds.

Proposition 1.4 (cf. [13])**.** *Let $(A, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ with $d = \dim A$. Let I be an $\mathfrak{m}$-primary ideal of A. Then*

$$\frac{e(I)}{d!} \le e_{\mathrm{HK}}(I) \le e(I).$$

Also, if $d \ge 2$, then the inequality in the left-hand side is strict; *see* [9].

We say that a local ring A is *unmixed* if $\dim \widehat{A}/\mathfrak{p} = \dim \widehat{A}$ holds for any associated prime ideal $\mathfrak{p}$ of $\widehat{A}$. The following theorem is an analogy of Nagata's theorem; see [20, (40.6)]. Furthermore, it is a starting point in this article.

Theorem 1.5 ([22, Theorem (1.5)])**.** *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of positive characteristic. Then A is regular if and only if $e_{\mathrm{HK}}(A) = 1$.*

It is not so easy to compute Hilbert-Kunz multiplicities in general. However, one has simple formulas for them in case of quotient singularities and in case of binomial hypersurfaces; see below or [4, Theorem 3.1].

Theorem 1.6 (cf. [22, Theorem (2.7)])**.** *Let $(A, \mathfrak{m}) \hookrightarrow (B, \mathfrak{n})$ be a module-finite extension of local domains. Then for every $\mathfrak{m}$-primary ideal I of A, we have*

$$e_{\mathrm{HK}}(I) = \frac{e_{\mathrm{HK}}(IB)}{[Q(B) : Q(A)]} \cdot [B/\mathfrak{n} : A/\mathfrak{m}],$$

where $Q(A)$ denotes the fraction field of A.

Now let us see some examples of Hilbert-Kunz multiplicities which are given by the above formula. First, we consider the Veronese subring A defined by

$$A = k[[X_1^{i_1} \cdots X_d^{i_d} \mid i_1, \ldots, i_d \ge 0, \sum i_j = r]].$$

Applying Theorem 1.6 to $A \hookrightarrow B = k[[x, y]]$, we get

$$e_{\mathrm{HK}}(A) = \frac{1}{r}\binom{d+r-1}{r}. \tag{1.1}$$

In particular, if $d = 2$, $r = e(A)$, then $e_{\mathrm{HK}}(A) = \frac{e(A)+1}{2}$.

Next, we consider the homogeneous coordinate ring of quadric hyperplanes in $\mathbb{P}_k^3$. Let k be a field of characteristic $p > 2$, and let R be the homogeneous coordinate ring of the hyperquadric Q defined by $q = q(X, Y, Z, W)$. Put $\mathfrak{M} = R_+$, the unique homogeneous maximal ideal of R, and $A = R_{\mathfrak{M}} \otimes_k \overline{k}$. By suitable coordinate transformation, we may assume that A is isomorphic to one of the following rings:

$$\begin{cases} k[[X, Y, Z, W]]/(X^2), & \text{if } \operatorname{rank}(q) = 1, \\ k[[X, Y, Z, W]]/(X^2 - YZ), & \text{if } \operatorname{rank}(q) = 2, \\ k[[X, Y, Z, W]]/(XY - ZW), & \text{if } \operatorname{rank}(q) = 3. \end{cases} \tag{1.2}$$

Then $e_{\mathrm{HK}}(A) = 2$, $\frac{3}{2}$, or $\frac{4}{3}$, respectively.

In order to state other important properties of Hilbert-Kunz multiplicities, the notion of tight closure is very important. See [11, 12, 13] for definition and the fundamental properties of tight closure. In particular, the notion of F-rational ring is essential in our argument.

Definition 1.7 ([6, 11, 12])**.** Let $(A, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$. We say that A is *weakly F-regular* (resp. *F-rational*) if every ideal (resp. every paramater ideal) is tightly closed. Also, A is *F-regular* (resp. *F-rational*) if any local ring of A is weakly F-regular (resp. F-rational).

Note that an F-rational local ring is normal and Cohen-Macaulay.

Hochster and Huneke have given the following criterion of tight closure in terms of Hilbert-Kunz multiplicity.

Theorem 1.8 (Length Criterion of Tight Closure (cf. [11, Theorem 8.17])). *Let $I \subseteq J$ be $\mathfrak{m}$-primary ideals of a local ring $(A, \mathfrak{m}, k)$ of characteristic $p > 0$.*

(1) *If $I^* = J^*$, then $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(J)$.*
(2) *Suppose that A is excellent, reduced and equidimensional. Then the converse of* (1) *is also true.*

The following theorem plays an important role in studying Hilbert-Kunz multiplicities for non-Cohen–Macaulay local rings.

Theorem 1.9 (Goto–Nakamura [8]). *Let $(A, \mathfrak{m}, k)$ be an equidimensional local ring which is a homomorphic image of a Cohen–Macaulay local ring of characteristic $p > 0$. Then*

(1) *If J is a parameter ideal of A, then $e(J) \geq l_A(A/J^*)$.*
(2) *Suppose that A is unmixed. If $e(J) = l_A(A/J^*)$, then A is F-rational* (*and thus*) *is Cohen–Macaulay.*

The next lemma is well-known in case of Cohen–Macaulay local rings (e.g. see [13]).

Corollary 1.10. *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of characteristic $p > 0$. Suppose that $e(A) = 2$. Then $\widehat{A}$ is F-rational if and only if $e_{\mathrm{HK}}(A) < 2$. When this is the case, A is a hypersurface.*

Proof. Since any Cohen–Macaulay local ring of multiplicity 2 is a hypersurface, it suffices to prove the first statement.

We may assume that A is complete and k is infinite. We can take a minimal reduction J of $\mathfrak{m}$. First, suppose that $e_{\mathrm{HK}}(A) < 2$. Then we show that A is Cohen–Macaulay, F-rational. By Goto–Nakamura's theorem, we have $2 = e(J) \geq l_A(A/J^*)$. If equality does not hold, $l_A(A/J^*) = 1$, that is, $J^* = \mathfrak{m}$. Then $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(J^*) = e_{\mathrm{HK}}(J) = e(J) = 2$ by Proposition 1.2. This is a contradiction. Hence $e(J) = l_A(A/J^*)$. By Goto–Nakamura's theorem again, we obtain that A is Cohen–Macaulay, F-rational.

Conversely, suppose that A is complete F-rational. Then since A is Cohen–Macaulay and $J^* = J \neq \mathfrak{m}$, we have $e_{\mathrm{HK}}(A) < e_{\mathrm{HK}}(J) = e(J) = 2$ by Length Criterion for Tight Closure. □

The next question is open in general. However, we will show that it is true for $\dim A \leq 3$; see Section 2.

Question 1.11. *If A is an unmixed local ring with $e_{\mathrm{HK}}(A) < 2$, then is it F-rational?*

2. Estimate of Hilbert-Kunz multiplicity

In this section, we will prove the key result to find a lower bound of Hilbert-Kunz multiplicities. Actually, it is a refinement of the argument which appeared in [22, Section 5] or in [23, Section 2]. The point is to use the tight closure J^* instead of "a parameter ideal J itself". This enables us to investigate Hilbert-Kunz multiplicities

of non-Cohen–Macaulay local rings. In Sections 3, 4, we will apply our method to unmixed local rings with $\dim A = 3,\ 4$.

Before stating our theorem, we introduce the following notation: For any positive real number s, we put

$$v_s := \mathrm{vol}\left\{ (x_1, \ldots, x_d) \in [0,1]^d \;\middle|\; \sum_{i=1}^{d} x_i \le s \right\}, \quad v'_s := 1 - v_s,$$

where $\mathrm{vol}\,(W)$ denotes the volume of $W \subseteq \mathbb{R}^d$. Then it is easy to see the following fact.

Fact 2.1. *Let $d \ge 1$ be an integer, and let s be a positive real number. Using the same notation as above, we have*

(1) $v_s + v'_s = 1$.
(2) $v'_{d-s} = v_s$.
(3) $v_{d/2} = v'_{d/2} = \frac{1}{2}$.
(4) *If $0 \le s \le 1$, then $v_s = \frac{s^d}{d!}$.*

Using the above notaion, a key result in this paper can be written as follows:

Theorem 2.2. *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of characteristic $p > 0$. Put $d = \dim A \ge 1$. Let J be a minimal reduction of $\mathfrak{m}$, and let r be an integer with $r \ge \mu_A(\mathfrak{m}/J^*)$, where J^* denotes the tight closure of J. Also, let $s \ge 1$ be a rational number. Then we have*

$$e_{\mathrm{HK}}(A) \ge e(A) \left\{ v_s - r \cdot \frac{(s-1)^d}{d!} \right\}. \tag{2.1}$$

Remark 3. When $1 \le s \le 2$, the right-hand side in Equation (2.1) is equal to $e(A)(v_s - r \cdot v_{s-1})$.

Before proving the theorem, we need the following lemma. In what follows, for any positive real number α, we define $I^\alpha := I^n$, where n is the minimum integer which does not exceed α.

Lemma 2.3. *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of characteristic $p > 0$ with $\dim A = d \ge 1$. Let J be a parameter ideal of A. Using the same notation as above, we have*

$$\lim_{q \to \infty} \frac{l_A(A/J^{sq})}{q^d} = \frac{e(J)s^d}{d!}, \qquad \lim_{q \to \infty} l_A\left(\frac{J^{sq} + J^{[q]}}{J^{[q]}} \right) = e(J) \cdot v'_s.$$

Proof. We may assume that A is complete. Let $x_1, \ldots, x_d$ be a system of parameters which generates J, and put $R := k[[x_1, \ldots, x_d]]$, $\mathfrak{n} = (x_1, \ldots, x_d)R$. Then R is a complete regular local ring and A is a finitely generated R-module with $A/\mathfrak{m} = R/\mathfrak{n}$. Since the assertion is clear in case of regular local rings, it suffices to show the following claim.

Claim: Let $\mathcal{I} = \{I_q\}_{q=p^e}$ be a set of ideals of A which satisfies the following conditions:

(1) For each $q = p^e$, $I_q = J_q A$ holds for some ideal $J_q \subseteq R$.
(2) There exists a positive integer t such that $\mathfrak{n}^{tq} \subseteq J_q$ for all $q = p^e$.
(3) $\lim_{q \to \infty} l_R(R/J_q)/q^d$ exists.

Then

$$\lim_{q\to\infty} \frac{l_A(A/I_q)}{q^d} = e(J)\cdot \lim_{q\to\infty} \frac{l_R(R/J_q)}{q^d}.$$

In fact, since A is unmixed, it is a torsion-free R-module of rank $e := e(J)$. Take a free R-module F of rank e such that $A_W \cong F_W$, where $W = A \setminus \{0\}$. Since F and A are both torsion-free, there exist the following short exact sequences of finitely generated R-modules:

$$0 \to F \to A \to C_1 \to 0, \qquad 0 \to A \to F \to C_2 \to 0,$$

where $(C_1)_W = (C_2)_W = 0$. In particular, $\dim C_1 < d$ and $\dim C_2 < d$.

Applying the tensor product $-\otimes_R R/J_q$ to the above two exact sequences, respectively, we get

$$\begin{aligned} l_R(A/I_q) &\le l_R(F/J_qF) + l_R(C_1/J_qC_1), \\ l_R(F/J_qF) &\le l_R(A/I_q) + l_R(C_2/J_qC_2). \end{aligned}$$

In general, if $\dim_R C < d$, then

$$\frac{l_R(C/J_qC)}{q^d} \le \frac{l_R(C/\mathfrak{n}^{tq}C)}{q^d} \to 0 \quad (q\to\infty).$$

Thus the required assertion easily follows from the above observation. $\square$

Proof of Theorem 2.2. For simplicity, we put $L = J^*$. We will give an upper bound of $l_A(\mathfrak{m}^{[q]}/J^{[q]})$. First, we have the following inequality:

$$\begin{aligned} l_A(\mathfrak{m}^{[q]}/J^{[q]}) &\le l_A\left(\frac{\mathfrak{m}^{[q]}+\mathfrak{m}^{sq}}{J^{[q]}}\right) \\ &= l_A\left(\frac{\mathfrak{m}^{[q]}+\mathfrak{m}^{sq}}{L^{[q]}+\mathfrak{m}^{sq}}\right) + l_A\left(\frac{L^{[q]}+\mathfrak{m}^{sq}}{L^{[q]}+J^{sq}}\right) \\ &\quad + l_A\left(\frac{L^{[q]}+J^{sq}}{J^{[q]}+J^{sq}}\right) + l_A\left(\frac{J^{[q]}+J^{sq}}{J^{[q]}}\right) =: \ell_1+\ell_2+\ell_3+\ell_4. \end{aligned}$$

Next, we see that $\ell_1 \le r\cdot l_A(A/J^{(s-1)q}) + O(q^{d-1})$. By our assumption, we can write as $\mathfrak{m} = L + Aa_1 + \cdots + Aa_r$. Since $\mathfrak{m}^{(s-1)q}a_i^q \subseteq \mathfrak{m}^{sq} \subseteq \mathfrak{m}^{sq} + L^{[q]}$, we have

$$\begin{aligned} \ell_1 = l_A\left(\frac{\mathfrak{m}^{[q]}+\mathfrak{m}^{sq}}{L^{[q]}+\mathfrak{m}^{sq}}\right) &\le \sum_{i=1}^{r} l_A\left(\frac{Aa_i^q + L^{[q]}+\mathfrak{m}^{sq}}{L^{[q]}+\mathfrak{m}^{sq}}\right) \\ &= \sum_{i=1}^{r} l_A\left(A/(L^{[q]}+\mathfrak{m}^{sq}) : a_i^q\right) \\ &\le r\cdot l_A(A/\mathfrak{m}^{(s-1)q}). \end{aligned}$$

Since J is a minimal reduction of $\mathfrak{m}$, we have $l_A(\mathfrak{m}^{tq}/J^{tq}) = O(q^{d-1})$. Thus we have the required inequality. Similarly, we get

$$\ell_2 = l_A\left(\frac{L^{[q]}+\mathfrak{m}^{sq}}{L^{[q]}+J^{sq}}\right) \le l_A(\mathfrak{m}^{sq}/J^{sq}) = O(q^{d-1}).$$

Also, we have $l_A(L^{[q]}/J^{[q]}) = O(q^{d-1})$ by Length Criterion for Tight Closure. Hence

$$l_A(\mathfrak{m}^{[q]}/J^{[q]}) \le r\cdot l_A(A/J^{(s-1)q}) + l_A\left(\frac{J^{[q]}+J^{sq}}{J^{[q]}}\right) + O(q^{d-1}).$$

It follows from the above argument that

$$\begin{aligned} e_{\mathrm{HK}}(J) - e_{\mathrm{HK}}(\mathfrak{m}) &\leq r \cdot \lim_{q\to\infty} \frac{l_A(A/J^{(s-1)q})}{q^d} + \lim_{q\to\infty} \frac{1}{q^d} l_A\left(\frac{J^{[q]}+J^{sq}}{J^{[q]}}\right) \\ &= r \cdot e \cdot \frac{(s-1)^d}{d!} + e \cdot v'_s. \end{aligned}$$

Since $e_{\mathrm{HK}}(J) = e(J) = e$, $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(\mathfrak{m})$ and $v'_s = 1 - v_s$, we get the required inequality. $\square$

The following fact is known, which gives a lower bound on Hilbert-Kunz multiplicities for hypersurface local rings.

Fact 2.4 (cf. [1, 2, 22]). *Let $(A, \mathfrak{m}, k)$ be a hypersurface local ring of characteristic $p > 0$ with $d = \dim A \geq 1$. Then*

$$e_{\mathrm{HK}}(A) \geq \beta_{d+1} \cdot e(A),$$

where β_{d+1} is given by the following equivalent formulas:

(a) $\displaystyle \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin\theta}{\theta}\right)^{d+1} d\theta;$

(b) $\displaystyle \frac{1}{2^d d!}\sum_{\ell=0}^{[\frac{d}{2}]}(-1)^\ell(d+1-2\ell)^d\binom{d+1}{\ell};$

(c) $\displaystyle \mathrm{vol}\left\{\underline{x}\in[0,1]^d \;\middle|\; \frac{d-1}{2} \leq \sum x_i \leq \frac{d+1}{2}\right\} = 1 - v_{\frac{d-1}{2}} - v'_{\frac{d+1}{2}}.$

TABLE 1.

d	0	1	2	3	4	5	6
β_{d+1}	1	1	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{115}{192}$	$\frac{11}{20}$	$\frac{5633}{11520}$

Remark 4. The above inequality is *not* best possible in general. In case of $d \geq 4$, one cannot prove the formula in the above fact as a corollary of our theorem. See also Proposition 3.10 and Theorem 4.3.

Using Theorem 2.2 and Lemma 3.3 of the next section, one can prove the following corollary, which has been already proved in [23] in the case of Cohen–Macaulay local rings.

Corollary 2.5 (cf. [23]). *Let $(A, \mathfrak{m}, k)$ be a two-dimensional unmixed local ring of characeristic $p > 0$. Put $e = e(A)$. Then*

$$e_{\mathrm{HK}}(A) \geq \frac{e+1}{2}. \tag{2.2}$$

Also, suppose $k = \overline{k}$. Then equality holds if and only if $\mathrm{gr}_{\mathfrak{m}}(A)$ is isomorphic to the Veronese subring $k[X,Y]^{(e)} = k[X^e, X^{e-1}Y, \ldots, XY^{e-1}, Y^e]$.

Moreover, if A is not F-rational, then we have

$$e_{\mathrm{HK}}(A) \geq \frac{e^2}{2(e-1)}.$$

Example 2.6 ([7, Corollary 3.19])**.** Let E be an elliptic curve over a field $k = \overline{k}$ of characteristic $p > 0$, and let $\mathcal{L}$ be a very ample line bundle on E of degree $e \geq 3$. Let R be the homogeneous coodinate ring (the section ring of $\mathcal{L}$) defined by

$$R = \bigoplus_{n \geq 0} H^0(E, \mathcal{L}^{\otimes n}).$$

Also, put $A = R_{\mathfrak{M}}$, where $\mathfrak{M}$ be the unique homogeneous maximal ideal of R. Then we have $e_{\mathrm{HK}}(A) = \frac{e^2}{2(e-1)}$.

3. Lower bounds in the case of three-dimensional local rings

In this section, we prove the following main theorem in this paper, which gives a lower bound on Hilbert-Kunz multiplicities for non-regular unmixed local rings of dimension 3.

Theorem 3.1. *Let $(A, \mathfrak{m}, k)$ be a three-dimensional unmixed local ring of characteristic $p > 0$. Then*

(1) *If A is not regular, then $e_{\mathrm{HK}}(A) \geq \frac{4}{3}$.*
(2) *Suppose that $k = \overline{k}$ and $\operatorname{char} k \neq 2$. Then the following conditions are equivalent*:
  (a) $e_{\mathrm{HK}}(A) = \frac{4}{3}$.
  (b) $\widehat{A} \cong k[[X, Y, Z, W]]/(X^2 + Y^2 + Z^2 + W^2)$.
  (c) $\mathrm{gr}_{\mathfrak{m}}(A) \cong k[X, Y, Z, W]/(X^2 + Y^2 + Z^2 + W^2)$. *That is,* $\mathrm{gr}_{\mathfrak{m}}(A) \cong k[X, Y, Z, W]/(XY - ZW)$.

Proposition 3.2. *Let $(A, \mathfrak{m}, k)$ be a three-dimensional unmixed local ring of characteristic $p > 0$. If $e_{\mathrm{HK}}(A) < 2$, then A is F-rational.*

From now on, we divide the proof of Theorem 3.1 and Proposition 3.2 into several steps. The following lemma is an analogy of Sally's theorem: If A is a Cohen–Macaulay local ring, then $\mu_A(\mathfrak{m}/J)(= \mu_A(\mathfrak{m}) - \dim A) \leq e(A) - 1$.

Lemma 3.3. *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of positive characteristic, and let J be a minimal reduction of $\mathfrak{m}$. Then*

(1) $\mu_A(\mathfrak{m}/J^*) \leq e(A) - 1$.
(2) *If A is not F-rational, then $\mu_A(\mathfrak{m}/J^*) \leq e(A) - 2$.*

Proof. We may assume that A is complete and thus is a homomorphic image of a Cohen–Macalay local ring. Put $e = e(A)$.

(1) By Theorem 1.9, we have that $\mu_A(\mathfrak{m}/J^*) \leq l_A(\mathfrak{m}/J^*) \leq e(J) - 1 = e - 1$.

(2) If A is not F-rational, then $l_A(A/J^*) \leq e(J) - 1 = e - 1$. Thus $\mu_A(\mathfrak{m}/J^*) \leq e - 2$, as required. □

Let A be an unmixed local ring which is not regular. Put $e = e(A)$, the multiplicity of A. Then e is an integer with $e \geq 2$. Thus the assertion (1) of Theorem 3.1 follows from the following lemma. Also, this implies that if $e_{\mathrm{HK}}(A) = \frac{4}{3}$ then $e(A) = 2$ without extra assumption.

Lemma 3.4. *Using the same notation as in Theorem 3.1, we put $e = e(A)$, the multiplicity of A.*

(1) *If* $e \geq 5$, *then* $e_{\mathrm{HK}}(A) > 2$
(2) *If* $e = 4$, *then* $e_{\mathrm{HK}}(A) \geq \frac{7}{4} > \frac{4}{3}$.
(3) *If* $e = 3$, *then* $e_{\mathrm{HK}}(A) \geq \frac{13}{8} > \frac{4}{3}$.
(4) *If* $e = 2$, *then* $e_{\mathrm{HK}}(A) \geq \frac{4}{3}$.

Remark 5. The lower bounds of $e_{\mathrm{HK}}(A)$ in Lemma 3.4 are not best possible.

Proof. We may assume that A is complete and k is infinite. By Lemma 3.3(1), we can apply Theorem 2.2 with $r = e - 1$. Namely, if $1 \leq s \leq 2$, then

$$e_{\mathrm{HK}}(A) \geq e(v_s - (e-1)v_{s-1}) = e\left(\frac{s^3}{6} - (e+2)\frac{(s-1)^3}{6}\right). \tag{3.1}$$

Define the real-valued function $f_e(s)$ by the right-hand side of Eq. (3.1). Then one can easily calculate $\max_{1\leq s\leq 2} f_e(s)$. In fact, if $e \geq 2$, then

$$\max_{1\leq s\leq 2} f_e(s) = f\left(\frac{e+2+\sqrt{e+2}}{e+1}\right) = \frac{e}{6}\left(\frac{e+2+\sqrt{e+2}}{e+1}\right)^2.$$

But, in order to prove the lemma, it is enough to use the following values only:

s	$\frac{3}{2}$	$\frac{7}{4}$	2
$f_e(s)$	$\frac{e(25-e)}{48}$	$\frac{e(289-27e)}{384}$	$\frac{e(6-e)}{6}$

(1) We show that $e_{\mathrm{HK}}(A) > 2$ if $e \geq 5$. If $e \geq 13$, then by Proposition 1.4,

$$e_{\mathrm{HK}}(A) \geq \frac{e}{3!} \geq \frac{13}{6} > 2.$$

So we may assume that $5 \leq e \leq 12$. Applying Eq. (3.1) for $s = \frac{3}{2}$, we get

$$e_{\mathrm{HK}}(A) \geq \frac{e(25-e)}{48} \geq \frac{5(25-5)}{48} = \frac{25}{12} > 2.$$

(2) Suppose that $e = 4$. Actually, applying Eq. (3.1) for $s = \frac{3}{2}$, we get

$$e_{\mathrm{HK}}(A) \geq \frac{e(25-e)}{48} = \frac{7}{4}.$$

(3) Suppose that $e = 3$. Applying Eq. (3.1) for $s = \frac{7}{4}$, we get

$$e_{\mathrm{HK}}(A) \geq \frac{e(289-27e)}{384} = \frac{13}{8}.$$

(4) Suppose that $e = 2$. Applying Eq. (3.1) for $s = 2$,

$$e_{\mathrm{HK}}(A) \geq \frac{e(6-e)}{6} = \frac{4}{3},$$

as required. $\square$

Before proving the second assertion of Theorem 3.1, we prove Proposition 3.2. For that purpose, we now focus three-dimensional non-F-rational local rings.

Let A be an unmixed local ring of positive characteristic with $\dim A = 3$. Put $e = e(A) \geq 2$. Also, suppose that A is not F-rational. If $e = 2$, then $e_{\mathrm{HK}}(A) = 2$ by Lemma 1.10. On the other hand, if $e \geq 5$, then $e_{\mathrm{HK}}(A) > 2$ by Lemma 3.4. Thus

in order to prove Proposition 3.2, it is enough to investigate the cases of $e = 3, 4$. Namely, Proposition 3.2 follows from the following lemma.

Lemma 3.5. *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of positive characteristic with $\dim A = 3$. Put $e = e(A)$. Suppose that A is not F-rational. Then*

(1) *If $e = 3$, then $e_{\mathrm{HK}}(A) \geq 2$.*
(2) *If $e = 4$, then $e_{\mathrm{HK}}(A) > 2$.*

Proof. We may assume that k is infinite. Then one can take a minimal reduction (say J) of $\mathfrak{m}$. By Lemma 3.3(2), we can apply Theorem 2.2 for $r = e - 2$. Thus if $1 \leq s \leq 2$, then

$$e_{\mathrm{HK}}(A) \geq e\left(\frac{s^3}{6} - (e+1)\frac{(s-1)^3}{6}\right). \tag{3.2}$$

(1) Suppose that $e = 3$. Applying Eq. (3.2) for $s = 2$, we get

$$e_{\mathrm{HK}}(A) \geq \frac{e(7-e)}{6} = 2.$$

(2) Suppose that $e = 4$. Applying Eq. (3.2) for $s = \frac{7}{4}$, we get

$$e_{\mathrm{HK}}(A) \geq \frac{e(316 - 27e)}{384} = \frac{13}{6} > 2,$$

as required. $\square$

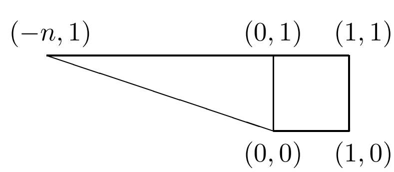
Example 3.6. Let $R = k[T, xT, xyT, yT, x^{-1}yT, x^{-2}yT, \ldots, x^{-n}yT]$ be a rational normal scroll and put $\mathfrak{m} = (T, xT, xyT, yT, x^{-1}yT, \ldots, x^{-n}yT)$. Then $A = R_{\mathfrak{m}}$ is a three-dimensional Cohen–Macaulay F-rational local domain with $e(A) = n + 2$, and

$$e_{\mathrm{HK}}(A) = \frac{e(A)}{2} + \frac{e(A)}{6(n+1)}.$$

Proof. Let $\mathcal{P} \subseteq \mathbb{R}$ be a convex polytope with vertex set

$$\Gamma = \{(0,0), (1,0), (1,1), (0,1), (-1,1), \ldots, (-n,1)\},$$

and put $\widetilde{\mathcal{P}} := \left\{(\alpha, 1) \in \mathbb{R}^3 \,|\, \alpha \in \mathcal{P}\right\}$ and $d\mathcal{P} := \left\{d \cdot \alpha \,|\, \alpha \in \mathcal{P}\right\}$ for every integer $d \geq 0$. Also, if we define a cone $\mathcal{C} = \mathcal{C}(\widetilde{\mathcal{P}}) := \{r\beta \,|\, \beta \in \widetilde{\mathcal{P}},\, 0 \leq r \in \mathbb{Q}\}$ and regard R as a homogeneous k-algebra with $\deg x = \deg y = 0$ and $\deg T = 1$, then the basis of R_d corresponds to the set $\{(\alpha, d) \in \mathbb{Z}^3 \,|\, \alpha \in \mathbb{Z}^2 \cap d\mathcal{P}\} = \{(\alpha, d) \in \mathbb{Z}^3 \,|\, \alpha \in \mathbb{Z}^2\} \cap \mathcal{C}$.

$\mathcal{P} =$ 

If we put $\Gamma_q = \{(0,0),(q,0),(q,q),(0,q),(-q,q),\ldots,(-nq,q)\}$, then $\mathfrak{m}^{[q]} = (x^a y^b T^q \,|\, (a,b) \in \Gamma_q)$. Since $[\mathfrak{m}^{[q]}]_d = \sum_{(a,b)\in\Gamma_q} R_{d-q}\, x^a y^b T^q$, we have

$$\begin{aligned} e_{\mathrm{HK}}(A) &= \lim_{q\to\infty} \frac{1}{q^3}\, l_A(A/\mathfrak{m}^{[q]}) \\ &= \lim_{q\to\infty} \frac{1}{q^3}\, \#\Big\{\mathbb{Z}^3 \cap \big(\mathcal{C} \setminus \bigcup_{(a,b)\in\Gamma_q} (a,b)+\mathcal{C}\big)\Big\}, \end{aligned}$$

that is,

$$e_{\mathrm{HK}}(A) = \lim_{q\to\infty} \frac{1}{q^3} \left[\sum_{d=0}^{\infty} \#\Big\{\mathbb{Z}^2 \cap \Big(d\mathcal{P} \setminus \bigcup_{(a,b)\in\Gamma_q} (a,b) + \max\{0, d-q\}\mathcal{P}\Big)\Big\}\right].$$

Also, we define a real continuous function $f : [0,\infty) \to \mathbb{R}$ by

$$f(t) = \text{ the volume of } \left[t\mathcal{P} \setminus \bigcup_{(a,b)\in\Gamma} (a,b) + \max\{0, t-1\}\mathcal{P}\right] \text{ in } \mathbb{R}^2,$$

then $e_{\mathrm{HK}}(A) = \displaystyle\int_0^{\infty} f(t)\,dt$. Let us denote the volume of $M \subseteq \mathbb{R}^2$ by $\mathrm{vol}\,(M)$. To calculate $e_{\mathrm{HK}}(A)$, we need to determine $f(t)$. Namely, we need to show the following claim.

Claim:

$$f(t) = \begin{cases} \mathrm{vol}\,(t\mathcal{P}) & (0 \le t < 1) \\ \mathrm{vol}\,(t\mathcal{P}) - (n+4)\mathrm{vol}\,((t-1)\mathcal{P}) & \left(1 \le t < \frac{n+2}{n+1}\right) \\ \frac{(n+2)t(2-t)}{2} + (n+2)\frac{(2-t)^2}{2n} & \left(\frac{n+2}{n+1} \le t < 2\right) \\ 0 & (t \ge 2) \end{cases}$$

To prove the claim, we may assume that $t \ge 1$. For simplicity, we put $M_{a,b} = (a,b) + (t-1)\mathcal{P}$ for every point $(a,b) \in \Gamma$. First suppose that $1 \le t < \frac{n+2}{n+1}$. Then since $1 - n(t-1) > t-1$, $M_{0,0} \cap M_{1,0} = \emptyset$. Similarly, one can easily see that any two $M_{a,b}$ do not intersect each other. Thus $f(t) = \mathrm{vol}\,(t\mathcal{P}) - (n+4)\mathrm{vol}\,((t-1)\mathcal{P})$. Next suppose that $\frac{n+2}{n+1} \le t < 2$. Then $\mathcal{P} \cap \{(x,y) \in \mathbb{R}^2 \,|\, 0 \le y \le t-1\} = M_{0,0} \cup M_{1,0} \cup T_0$, where T_0 is a triangle with vertex $(t-1, 0)$, $(1,0)$ and $\left(t-1, \frac{2-t}{n}\right)$. Similarly, there exist $(n+1)$ triangles $T_1, \ldots, T_{n+1}$ having the same volumes as T_0 such that

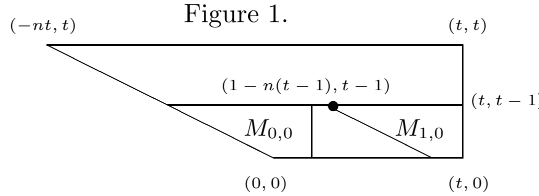
$$\mathcal{P} \cap \{(x,y) \in \mathbb{R}^2 \,|\, 1 \le y \le t\} = M_{-n,1} \cup \cdots \cup M_{1,1} \cup M_{0,1} \cup M_{1,1} \cup T_1 \cup \cdots \cup T_{n+1}$$

and any two T_i's do not intersect each other. Thus

$$\begin{aligned} f(t) &= \mathrm{vol}\,(\mathcal{P} \cap \{(x,y) \in \mathbb{R}^2 \,|\, t-1 \le y \le 1\}) + (n+2)\mathrm{vol}\,(T_0) \\ &= \frac{(n+2)t(2-t)}{2} + (n+2)\frac{(2-t)^2}{2n}. \end{aligned}$$

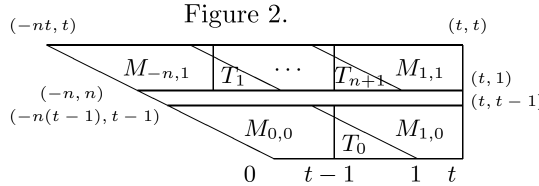
Finally, suppose that $t \ge 2$. Then since $\mathcal{P}$ is covered by $M_{a,b}$'s, we have $f(t) = 0$, as required.

Figure 1.



The case where $1 \le t < \frac{n+2}{n+1}$

Figure 2.



The case where $\frac{n+2}{n+1} \le t < 2$

Using the above claim, let us calculate $e_{\mathrm{HK}}(A)$. Note that $\mathrm{vol}\,(t\mathcal{P}) = \frac{(n+2)t^2}{2}$.

$$
\begin{aligned}
e_{\mathrm{HK}}(A) &= \int_0^{\frac{n+2}{n+1}} \frac{(n+2)t^2}{2}\,dt - (n+4)\int_1^{\frac{n+2}{n+1}} \frac{(n+2)(t-1)^2}{2}\,dt \\
&\quad + \int_{\frac{n+2}{n+1}}^2 \frac{(n+2)t(2-t)}{2}\,dt + (n+2)\int_{\frac{n+2}{n+1}}^2 \frac{(2-t)^2}{2n}\,dt \\
&= (n+2)\left[\frac{1}{2} + \frac{1}{6(n+1)}\right],
\end{aligned}
$$

as required. $\square$

Discussion 3.7. Let $(A, \mathfrak{m}, k)$ be a complete unmixed local ring of positive characteristic with $\dim A = 3$ and $e := e(A) = 3$. What is the smallest value of $e_{\mathrm{HK}}(A)$ among such rings?

The function $f_e(s) = 3\left(\frac{s^3}{6} - 5\frac{(s-1)^3}{6}\right)$, which appeared in Eq. (3.1), takes a maximal value

$$f\left(\frac{5+\sqrt{5}}{4}\right) = \frac{15+5\sqrt{5}}{16} = 1.636\ldots$$

in $s \in [1, 2]$. Hence $e_{\mathrm{HK}}(A) \ge 1.636\ldots$. But we beleive that this is not best possible.

Suppose that A is complete and $e_{\mathrm{HK}}(A) < 2$. Then A is F-rational by Lemma 3.5. Thus it is Cohen–Macaulay and $3+1 \le v = \mathrm{emb}\,(A) \le d+e-1 = 3+3-1 = 5$. If $v \ne 5$, then A is a hypersurface and $e_{\mathrm{HK}}(A) \ge \frac{2}{3} \cdot e = 2$ by Fact 2.4. Hence we may assume that $v = 5$, that is, A has a maximal embedding dimension. If we write as $A = R/I$, where R is a complete regular local ring with $\dim R = 5$, then height $I = 2$. By Hilbert–Burch's theorem, there exists a 2×3-matrix $\mathbb{M}$ such that $I = I_2(\mathbb{M})$, the ideal generated by all 2-minors of $\mathbb{M}$. In particular, A can be written as $A = B/aB$, where $B = k[X]/I_2(X)$, X is a generic 2×3-matrix and a is a prime element of B. This implies that

$$e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(B/aB) \ge e_{\mathrm{HK}}(B) = 3\left\{\frac{1}{2} + \frac{1}{4!}\right\} = \frac{13}{8} = 1.625;$$

see [5, Section 3].

For example, if $A = k[[T, xT, xyt, yT, x^{-1}yT]]$ is a rational normal scroll, then $e_{\mathrm{HK}}(A) = \frac{7}{4} = 1.75$ by Example 3.6. Is this the smallest value?

Discussion 3.8. Let $(A, \mathfrak{m}, k)$ be a complete unmixed local ring of positive characteristic. Suppose that $\dim A = 3$ and $e(A) = 4$. What is the smallest value of $e_{\mathrm{HK}}(A)$ among such rings?

As in Discussion 3.7, it suffices to consider F-rational local rings only. For example, let $A = k[[x, y, z]]^{(2)}$ be the Veronese subring. Then A is an F-rational local domain with $e(A) = 4$ and $e_{\mathrm{HK}}(A) = 2$. Also, let A be the completion of the Rees algebra $R(\mathfrak{n})$ over an F-rational double point $(R, \mathfrak{n})$ of dimension 2. Then A is an F-rational local domain with $e(A) = 4$ and $e_{\mathrm{HK}}(A) \geq 2$ (we beleive that this inequality is strict).

On the other hand, the function $f_e(s)$ which appeared in Eq. (3.1), takes a maximal value

$$f\left(\frac{6+\sqrt{6}}{5}\right) = \frac{28 + 8\sqrt{6}}{25} = 1.903\ldots$$

in $s \in [1, 2]$. Hence the fact that we can prove now is "$e_{\mathrm{HK}}(A) \geq 1.903\ldots$" only.

Based on Corollary 2.5 and Discussion 3.8, we pose the following conjecture.

Conjecture 3.9. *Let A be a complete unmixed local ring of positive characteristic with $\dim A = 3$. Let r be an integer. If $e(A) = r^2$, then*

$$e_{\mathrm{HK}}(A) \geq \frac{(r+1)(r+2)}{6}.$$

Also, the equality holds if and only if A is isomorphic to $k[[x, y, z]]^{(r)}$.

In the rest of this section, we prove the second statement of Theorem 3.1. Let $(A, \mathfrak{m}, k)$ be a complete unmixed local ring of characteristic $p > 0$. If $e_{\mathrm{HK}}(A) = \frac{4}{3}$, then A is an F-rational hypersurface with $e(A) = 2$ by the above observation. Furthermore, suppose that $k = \overline{k}$ and $\operatorname{char} k \neq 2$. Then we may assume that A can be written as the form $k[[X, Y, Z, W]]/(X^2 - \varphi(Y, Z, W))$. To study Hilbert-Kunz multiplicities for these rings, we prove the improved version of Theorem 2.2.

Proposition 3.10. *Let k be an algebraically closed field of $\operatorname{char} k \neq 2$, and let $A = k[[X, Y, Z, W]]/(X^2 - \varphi(Y, Z, W))$ be an F-rational hypersurface local ring. Let a, b, c be integers with $2 \leq a \leq b \leq c$.*

Suppose that there exists a function $\operatorname{ord} : A \to \mathbb{Q}$ which satisfies the following conditions:

1. $\operatorname{ord}(\alpha) \geq 0$*; and* $\operatorname{ord}(\alpha) = \infty \iff \alpha = 0$.
2. $\operatorname{ord}(x) = 1/2$, $\operatorname{ord} y = 1/a$, $\operatorname{ord} z = 1/b$, *and* $\operatorname{ord} w = 1/c$.
3. $\operatorname{ord}(\varphi) \geq 1$.
4. $\operatorname{ord}(\alpha + \beta) \geq \min\{\operatorname{ord}(\alpha), \operatorname{ord}(\beta)\}$.
5. $\operatorname{ord}(\alpha\beta) \geq \operatorname{ord}(\alpha) + \operatorname{ord}(\beta)$.

Then we have

$$e_{\mathrm{HK}}(A) \geq 2 - \frac{abc}{12}(N^3 - n^3),$$

where

$$N = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{2}, \qquad n = \max\left\{0,\ N - \frac{2}{c}\right\}.$$

In particular, if $(a,b,c) \neq (2,2,2)$*, then* $e_{\mathrm{HK}}(A) > \frac{4}{3}$.

Proof. First, we define a filtration $\{F_n\}_{n \in \mathbb{Q}}$ as follows:

$$F_n := \{\alpha \in A \,|\, \mathrm{ord}\,(\alpha) \geq n\}.$$

Then every F_n is an ideal and $F_m F_n \subseteq F_{m+n}$ holds for all m, $n \in \mathbb{Q}$. Using F_n instead of $\mathfrak{m}^n$, we shall estimate $l_A(\mathfrak{m}^{[q]}/J^{[q]})$.

Set $J = (y,z,w)A$ and fix a sufficiently large power $q = p^e$. Put

$$s = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \qquad N = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{2}.$$

Since J is a minimal reduction of $\mathfrak{m}$ and $xy^{q-1}z^{q-1}w^{q-1}$ generates the socle of $A/J^{[q]}$, we have that $F_{sq} \subseteq J^{[q]}$. Also, since $B = A/J^{[q]}$ is an Artinian Gorenstein local ring, we get

$$F_{\frac{(N+1)q}{2}} B \subseteq 0 :_B F_{\frac{Nq}{2}} B \cong K_{B/F_{\frac{Nq}{2}}B}.$$

Hence, by Matlis duality theorem, we get

$$l_A\left(\frac{F_{\frac{(N+1)q}{2}} + J^{[q]}}{J^{[q]}}\right) \leq l_B\left(K_{B/F_{\frac{Nq}{2}}B}\right) = l_B\left(B/F_{\frac{Nq}{2}}B\right).$$

On the other hand, since $x^q \in F_{\frac{q}{2}}$ by the assumption, we have

$$x^q F_{\frac{Nq}{2}} \subseteq F_{\frac{(N+1)q}{2}}.$$

Therefore by the similar argument as in the proof of Theorem 2.2, we get

$$\begin{aligned}
l_A(\mathfrak{m}^{[q]}/J^{[q]}) &\leq l_A\left(\frac{Ax^q + J^{[q]} + F_{\frac{(N+1)q}{2}}}{F_{\frac{(N+1)q}{2}} + J^{[q]}}\right) + l_A\left(\frac{F_{\frac{(N+1)q}{2}} + J^{[q]}}{J^{[q]}}\right) \\
&\leq l_A\left(A/(J^{[q]} + F_{\frac{(N+1)q}{2}}) : x^q\right) + l_B\left(B/F_{\frac{Nq}{2}}B\right) \\
&\leq 2 \cdot l_A\left(A/J^{[q]} + F_{\frac{N}{2}q}\right).
\end{aligned}$$

In fact, since

$$\begin{aligned}
& \lim_{q\to\infty} \frac{1}{q^3}\, l_A\left(A/J^{[q]} + F_{\frac{Nq}{2}}\right) \\
= \;& e(A) \cdot \lim_{q\to\infty} \frac{1}{q^3} \;\; \mathrm{vol}\left\{(x,y,z) \in [0,q]^3 \;\middle|\; \frac{y}{a} + \frac{z}{b} + \frac{w}{c} \leq \frac{Nq}{2}\right\} \\
= \;& 2 \cdot \mathrm{vol}\left\{(x,y,z) \in [0,1]^3 \;\middle|\; \frac{y}{a} + \frac{z}{b} + \frac{w}{c} \leq \frac{N}{2}\right\} \\
= \;& \frac{abc}{24}(N^3 - n^3),
\end{aligned}$$

we get

$$e_{\mathrm{HK}}(A) \geq 2 - 2 \cdot \frac{abc}{24}(N^3 - n^3) = 2 - \frac{abc}{12}(N^3 - n^3),$$

as required. $\square$

Example 3.11. Let k be an algebraically closed field of $\mathrm{char}\, k \neq 2$. Put $A = k[[X,Y,Z,W]]/(f(X,Y,Z,W))$.

$$f(X,Y,Z,W) = X^2 + Y^3 + Z^3 + W^3 \quad \Longrightarrow \quad e_{\mathrm{HK}}(A) \geq \frac{55}{32};$$

$$f(X,Y,Z,W) = X^2 + Y^2 + Z^3 + W^3 \quad \Longrightarrow \quad e_{\mathrm{HK}}(A) \geq \frac{14}{9};$$

$$f(X,Y,Z,W) = X^2 + Y^2 + Z^2 + W^c \quad \Longrightarrow \quad e_{\mathrm{HK}}(A) \geq \frac{9c^2-4}{6c^2}.$$

Proof of Theorem 3.1(2). Put $G = \mathrm{gr}_{\mathfrak{m}}(A)$ and $\mathfrak{M} = \mathrm{gr}_{\mathfrak{m}}(A)_+$. The implication $(a) \Longrightarrow (b)$ follows from Proposition 3.10. $(b) \Longrightarrow (c)$ is clear. Suppose (c). Then $e_{\mathrm{HK}}(G_{\mathfrak{M}}) = \frac{4}{3}$. Also, by Proposition 1.3 and Theorem 3.1 (1), we have that $\frac{4}{3} \leq e_{\mathrm{HK}}(A) \leq e_{\mathrm{HK}}(G_{\mathfrak{M}}) = \frac{4}{3}$. Thus $e_{\mathrm{HK}}(A) = \frac{4}{3}$, as required. $\square$

Also, the following corollary follows from the proof of Proposition 3.10 and Example 3.11.

Corollary 3.12. *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of characteristic $p > 0$ with $\dim A = 3$. Assume that $k = \overline{k}$ and $p \neq 2$. Then the following conditions are equivalent:*

(1) $\frac{4}{3} < e_{\mathrm{HK}}(A) \leq \frac{3}{2}$.
(2) $\mathrm{gr}_{\mathfrak{m}}(A) \cong k[X,Y,Z]/(X^2+Y^2+Z^2)$.
(3) *A is isomorphic to a hypersurface $k[[X,Y,Z,W]]/(X^2+Y^2+Z^2+W^c)$ for some integer $c \geq 3$.*

When this is the case, $e_{\mathrm{HK}}(A) \geq \dfrac{3}{2} - \dfrac{2}{3c^2}$.

4. A generalization of the main result to higher dimensional case

In this section, we want to consider a generalization of Theorem 3.1 in case of $\dim A \geq 4$. Let $d \geq 1$ be an integer and $p > 2$ a prime number. If we put

$$A_{p,d} := \overline{\mathbb{F}_p}[[X_0, X_1, \ldots, X_d]]/(X_0^2 + \cdots + X_d^2),$$

then we guess that $e_{\mathrm{HK}}(A_{p,d}) = s_{\mathrm{HK}}(p,d)$ holds according to the observations until the previous section. In the following, let us formulate this as a conjecture and prove that it is also true in case of $\dim A = 4$.

In [10], Han and Monsky gave an algorism to calculate $e_{\mathrm{HK}}(A_{p,d})$, but it is not so easy to represent it as a quotient of two polynomials of p for any fixed $d \geq 1$.

d	1	2	3	4
$e_{\mathrm{HK}}(A_{p,d})$	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{29p^2+15}{24p^2+12}$

On the other hand, surprisingly, Monsky proved the following theorem:

Theorem 4.1 (Monsky [19]). *Under the above notation, we have*

$$\lim_{p \to \infty} e_{\mathrm{HK}}(A_{p,d}) = 1 + \frac{c_d}{d!}, \tag{4.1}$$

where

$$\sec x + \tan x = \sum_{d=0}^{\infty} \frac{c_d}{d!} x^d \quad \left(|x| < \frac{\pi}{2}\right). \tag{4.2}$$

Remark 6. It is known that the Taylor expansion of $\sec x$ (resp. $\tan x$) at origin can be written as follows:

$$\begin{aligned}\sec x &= \sum_{i=0}^{\infty} \frac{E_{2i}}{(2i)!} x^{2i}, \\ \tan x &= \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i}(2^{2i}-1)B_{2i}}{(2i)!} x^{2i-1},\end{aligned}$$

where E_{2i} (resp. B_{2i}) is said to be Euler number (resp. Bernoulli number).

Also, c_d appeared in Eq.(4.1) is a positive integer since $\cos t$ is an unit element in a ring $\mathcal{H} = \left\{\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \,|\, a_n \in \mathbb{Z} \text{ for all } n \geq 0\right\}$.

Based on the above observation, we pose the following conjecture.

Conjecture 4.2. *Let $d \geq 1$ be an integer and $p > 2$ a prime number. Put*

$$A_{p,d} := \overline{\mathbb{F}_p}[[X_0, X_1, \ldots, X_d]]/(X_0^2 + \cdots + X_d^2).$$

Let $(A, \mathfrak{m}, k)$ be a d-dimensional unmixed local ring with $k = \overline{\mathbb{F}_p}$. Then the following statements hold.

(1) *If A is not regular, then $e_{\mathrm{HK}}(A) \geq e_{\mathrm{HK}}(A_{p,d}) \geq 1 + \frac{c_d}{d!}$. In particular, $s_{\mathrm{HK}}(p, d) = e_{\mathrm{HK}}(A_{p,d})$.*

(2) *If $e_{\mathrm{HK}}(A) = e_{\mathrm{HK}}(A_{p,d})$, then $\widehat{A} \cong A_{p,d}$ as local rings.*

In the following, we prove that this is true in case of $\dim A = 4$. Note that

$$\lim_{p\to\infty} e_{\mathrm{HK}}(A_{p,4}) = \lim_{p\to\infty} \frac{29p^2+15}{24p^2+12} = \frac{29}{24} = 1 + \frac{c_4}{4!}.$$

Theorem 4.3. *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of characteristic $p > 0$ with $\dim A = 4$. If $e(A) \geq 3$, then $e_{\mathrm{HK}}(A) \geq \frac{5}{4} = \frac{30}{24}$.*

Suppose that $k = \overline{k}$ and $\operatorname{char} k \neq 2$. Put

$$A_{p,4} = \overline{\mathbb{F}_p}[[X_0, X_1, \ldots, X_4]]/(X_0^2 + \cdots + X_4^2).$$

Then the following statement holds.

(1) *If A is not regular, then*

$$e_{\mathrm{HK}}(A) \geq e_{\mathrm{HK}}(A_{p,4}) = \frac{29p^2+15}{24p^2+12}.$$

(2) *The following conditions are equivalent:*
   (a) *Equality holds in* (1).
   (b) $e_{\mathrm{HK}}(A) < \frac{5}{4}$.
   (c) *The completion of A is isomorphic to $A_{p,4}$.*

Proof. Put $e = e(A)$, the multiplicity of A. We may assume that A is complete with $e \geq 2$ and k is infinite. In particular, A is a homomorphic image of a Cohen–Macalay local ring, and there exists a minimal reduction J of $\mathfrak{m}$. Then $\mu_A(\mathfrak{m}/J^*) \leq e - 1$ by Lemma 3.3. We first show that $e_{\mathrm{HK}}(A) \geq \frac{5}{4}$ if $e \geq 3$.

Claim 1: If $3 \leq e \leq 10$, then $e_{\mathrm{HK}}(A) \geq \frac{5}{4}$.

Putting $r = e-1$ and $s = 2$ in Theorem 2.2, since $v_2 = \frac{1}{2}$, we have

$$e_{\mathrm{HK}}(A) \geq e\left\{v_2 - \frac{(e-1)1^4}{4!}\right\} = \frac{(13-e)e}{24} \geq \frac{30}{24},$$

as required.

Claim 2: If $11 \leq e \leq 29$, then $e_{\mathrm{HK}}(A) \geq \frac{737}{384}\left(> \frac{5}{4}\right)$.

By Fact 2.4, we have $v_{3/2} = \frac{1-\beta_{4+1}}{2} = \frac{77}{384}$. Putting $r = e-1$ and $s = \frac{3}{2}$ in Theorem 2.2, we have

$$e_{\mathrm{HK}}(A) \geq e\left\{v_{3/2} - \frac{e-1}{24}\cdot\left(\frac{1}{2}\right)^4\right\} = \frac{(78-e)e}{384} \geq \frac{11(78-11)}{384} = \frac{737}{384},$$

as required.

Claim 3: If $e \geq 30$, then $e_{\mathrm{HK}}(A) \geq \frac{5}{4}$.

By Proposition 1.4, we have $e_{\mathrm{HK}}(A) \geq \frac{e}{4!} \geq \frac{30}{24}$.

In the following, we assume that $k = \overline{k}$, $\operatorname{char} k \neq 2$ and $e \geq 2$. To see (1),(2), we may assume that $e = 2$ by the above argument. Then since $e_{\mathrm{HK}}(A) = 2$ if A is not F-rational, we may also assume that A is F-rational and thus is a hypersurface. Thus A can be written as the following form:

$$A = k[[X_0, X_1, \ldots, X_4]]/(X_0^2 - \varphi(X_1, X_2, X_3, X_4))$$

If A is isomorphic to $A_{p,4}$, then by [10], it is known that

$$e_{\mathrm{HK}}(A) = \frac{29p^2+15}{24p^2+12}.$$

Suppose that A is not isomorphic to $A_{p,4}$. Then one can take a minimal numbers of generators x, y, z, w, u of $\mathfrak{m}$ and one can define a function ord : $A \to \mathbb{Q}$ such that

$$\operatorname{ord}(x) = \operatorname{ord}(y) = \operatorname{ord}(z) = \operatorname{ord}(z) = \frac{1}{2}, \quad \operatorname{ord}(u) = \frac{1}{3}.$$

If we put $J = (y, z, w, u)A$ and $F_n = \{\alpha \in A \,|\, \operatorname{ord}(\alpha) \geq n\}$, then by the similar argument as in the proof of Proposition 3.10, we have

$$l_A(\mathfrak{m}^{[q]}/J^{[q]}) \leq 2 \cdot l_A(A/J^{[q]} + F_{2q/3}).$$

Divided the both-side by q^d and taking a limit $q \to \infty$, we get

$$e(A) - e_{\mathrm{HK}}(A) \leq 2\cdot e(A)\cdot \operatorname{vol}\left\{(y,z,w,u) \in [0,1]^4 \,\middle|\, \frac{y}{2} + \frac{z}{2} + \frac{w}{2} + \frac{u}{3} \leq \frac{2}{3}\right\}.$$

To calculate the volume in the right-hand side, we put

$$F_u = \begin{cases} \frac{1}{6}\left(\frac{4}{3} - \frac{2}{3}u\right)^3 - \frac{1}{6}\left(\frac{1}{3} - \frac{2}{3}u\right)^3 & (0 \leq u \leq \frac{1}{2}) \\ \frac{1}{6}\left(\frac{4}{3} - \frac{2}{3}u\right)^3 & (\frac{1}{2} \leq u \leq 1) \end{cases}$$

Then one can easily calculate

$$\text{the above volume} = \int_0^1 F_u \, du = \frac{237}{2^4 3^4}.$$

It follows that

$$e_{\mathrm{HK}}(A) \geq 2 - 4 \times \frac{237}{2^4 3^4} = \frac{411}{324} > \frac{5}{4}.$$

□

The following conjecture also holds if $\dim A \leq 4$.

Conjecture 4.4. *Under the same notation as in Conjecture 4.2, if* $e(A) \geq 3$*, then*

$$e_{\mathrm{HK}}(A) \geq \frac{c_d + 1}{d!}.$$

Discussion 4.5. Let $d \geq 2$ be an integer and fix a prime number $p \gg d$. Assume that Conjectures 4.2 and 4.4 are true. Also, assume that $s_{\mathrm{HK}}(p,d) < s_{\mathrm{HK}}(p,d-1)$ for all $d \geq 3$. Let $A = k[X_0,\ldots,X_v]/I$ be a d-dimensional homogeneous unmixed k-algebra with $\deg X_i = 1$, and let $\mathfrak{m}$ be a unique homogeneous maximal ideal of A. Suppose that $\operatorname{char} k = p > 2$ and $k = \overline{\mathbb{F}_p}$. Then $e_{\mathrm{HK}}(A) = s_{\mathrm{HK}}(p,d)$ implies that $\widehat{A_{\mathfrak{m}}} \cong A_{p,d}$.

In fact, if $e_{\mathrm{HK}}(A) = s_{\mathrm{HK}}(p,d)$, then we may assume that $e_{\mathrm{HK}}(A) \leq \frac{c_d+1}{d!}$. Thus $e(A_{\mathfrak{m}}) = 2$ by the assumption that Conjecture 4.4 is true. For any prime ideal $PA_{\mathfrak{m}}$ of $A_{\mathfrak{m}}$ such that $P \neq \mathfrak{m}$, we have $e_{\mathrm{HK}}(A_P) \leq e_{\mathrm{HK}}(A) = s_{\mathrm{HK}}(p,d) < e_{\mathrm{HK}}(p,n)$, where $n = \dim A_P < d$. Since A_P is also unmixed, we have that A_P is regular. Thus $A_{\mathfrak{m}}$ has an isolated singularity. Hence A is a non-degenerate quadric hyperplane. In other words, $\widehat{A_{\mathfrak{m}}}$ is isomorphic to $A_{p,d}$.

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